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WHAT NUMBERS COULD NOT BE[1]

The attention of the mathematician focuses primarily upon mathematical structure, and his intellectual delight arises (in part) from seeing that a given theory exhibits such and such a structure, from seeing how one structure is "modelled" in another, or in exhibiting some new structure and showing how it relates to previously studied ones . . . . But . . . the mathematician is satisfied so long as he has some "entities" or "objects" (or "sets" or "numbers" or "functions" or "spaces" or "points") to work with, and he does not inquire into their inner character or ontological status.

The philosophical logician, on the other hand, is more sensitive to matters of ontology and will be especially interested in the kind or kinds of entities there are actually . . . . He will not be satisfied with being told merely that such and such entities exhibit such and such a mathematical structure. He will wish to inquire more deeply into what these entities are, how they relate to other entities . . . . Also he will wish to ask whether the entity dealt with is *sui generis* or whether it is in some sense *reducible* to (or *constructible* in terms of) other, perhaps more fundamental entities.

—R. M. Martin, *Intension and Decision*

We can . . . by using . . . [our] . . . definitions say what is meant by
"the number 1 + 1 belongs to the concept F"
and then, using this, give the sense of the expression
"the number 1 + 1 + 1 belongs to the concept F"
and so on; but we can never . . . decide by means of our definitions whether any concept has the number Julius Caesar belonging to it, or whether that same familiar conqueror of Gaul is a number or not.

—G. Frege, *The Foundations of Arithmetic*

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I. The Education

Imagine Ernie and Johnny, sons of two militant logicists—children who have not been taught in the vulgar (old-fashioned) way but for whom the pedagogical order of things has been the epistemological order. They did not learn straight off how to to count. Instead of beginning their mathematical training with arithmetic as ordinary men know it, they first learned logic—in their case, actually set theory. Then they were told about the numbers. But to tell people in their position about the numbers was an easy task—very much like the one which faced Monsieur Jourdain's tutor (who, oddly enough, was a philosopher). The parents of our imagined children needed only to point out what aspect or part of what the children already knew, under other names, was what ordinary people called "numbers." Learning the numbers merely involved learning new names for familiar sets. Old (set-theoretic) truths took on new (number-theoretic) clothing.

The way in which this was done will, however, bear some scrutiny and re-examination. To facilitate the exposition, I will concentrate on Ernie and follow his arithmetical education to its completion. I will then return to Johnny.

It might have gone as follows. Ernie was told that there was a set whose members were what ordinary people referred to as the (natural) numbers, and that these were what he had known all along as the elements of the (infinite) set N. He was further told that there was a relation defined on these "numbers" (henceforth I shall usually omit the shudder quotes), the *less-than* relation, under which the numbers were well ordered. This relation, he learned, was really the one, defined on N, for which he had always used the letter "R." And indeed, speaking intuitively now, Ernie could verify that every nonempty subset of N contained a "least" element—that is, one that bore R to every other member of the subset. He could also show that nothing bore R to itself, and that R was transitive, antisymmetric, irreflexive, and connected in N. In short, the elements of N formed a progression, or series, under R.

And then there was 1, the smallest number (for reasons of future convenience we are ignoring 0). Ernie learned that what people had been referring to as 1 was really the element *a* of $\mathcal{N}$, the first, or least, element of $\mathcal{N}$ under *R*. Talk about "successors" (each number is said to have one) was easily translated in terms of the concept of the "next" member of $\mathcal{N}$ (under *R*). At this point, it was no trick to show that the assumptions made by ordinary mortals about numbers were in fact theorems for Ernie. For on the basis of his theory, he could establish Peano's axioms —an advantage which he enjoyed over ordinary mortals, who must more or less take them as given, or self-evident, or meaningless-but-useful, or what have you.[2]

There are two more things that Ernie had to learn before he could truly be said to be able to speak with the vulgar. It had to be pointed out to him which operations on the members of $\mathcal{N}$ were the ones referred to as "addition," "multiplication," "exponentiation," and so forth. And here again he was in a position of epistemological superiority. For whereas ordinary folk had to introduce such operations by recursive definition, a euphemism for postulation, he was in a position to show that these operations could be *explicity* defined. So the additional postulates assumed by the number people were also shown to be derivable in his theory, once it was seen which set-theoretic operations addition, multiplication, and so forth really are.

The last element needed to complete Ernie's education was the explanation of the *applications* of these devices: counting and measurement. For they employ concepts beyond those as yet introduced. But fortunately, Ernie was in a position to see what it was that he was doing that corresponded to these activities (we will concentrate on counting, assuming that measurement can be explained either similarly or in terms of counting).

There are two kinds of counting, corresponding to transitive and intransitive uses of the verb "to count." In one, "counting" admits of a direct object, as in "counting the marbles"; in the other it does not. The case I have in mind is that of the preoperative patient being prepared for the operating room. The ether mask is placed over his face and he is told to count, as far as he

[2] I will not bore the reader with the details of the proofs.

can. He has not been instructed to count anything at all. He has merely been told to count. A likely story is that we normally learn the first few numbers in connection with sets having that number of members—that is, in terms of *transitive* counting (thereby learning the use of numbers) and then learn how to generate "the rest" of the numbers. Actually, "the rest" always remains a relatively vague matter. Most of us simply learn that we will never run out, that our notation will extend as far as we will ever need to count. Learning these words, and how to repeat them in the right order, is learning *intransitive* counting. Learning their use as measures of sets is learning *transitive* counting. Whether we learn one kind of counting before the other is immaterial so far as the initial numbers are concerned. What is certain, and not immaterial, is that we will have to learn some recursive procedure for generating the *notation* in the proper order before we have learned to count transitively, for to do the latter is, either directly or indirectly, to correlate the elements of the number series with the members of the set we are counting. It seems, therefore, that it is possible for someone to learn to count intransitively without learning to count transitively. But not vice versa. This is, I think, a mildly significant point. But what *is* transitive counting, exactly?

To count the members of a set is to determine the cardinality of the set. It is to establish that a particular relation C obtains between the set and one of the numbers—that is, one of the elements of $\mathcal{N}$ (we will restrict ourselves to counting finite sets here). Practically speaking, and in simple cases, one determines that a set has k elements by taking (sometimes metaphorically) its elements one by one as we say the numbers one by one (starting with 1 and in order of magnitude, the last number we say being k). To count the elements of some k-membered set b is to establish a one-to-one correspondence between the elements of b and the elements of $\mathcal{N}$ less than or equal to k. The relation "pointing-to-each-member-of-b-in-turn-while-saying-the-numbers-up-to-and-including-k" establishes such a correspondence.

Since Ernie has at his disposal the machinery necessary to show of any two equivalent finite sets that such a correspondence exists between them, it will be a theorem of his system that any set has k members if and only if it can be put into one-to-one correspondence with the set of numbers less than or equal to k.[3]

Before Ernie's education (and the analysis of number) can be said to have been completed, one last condition on R should be mentioned: that R must be at least recursive, and possibly even primitive recursive. I have never seen this condition included in the analysis of number, but it seems to me so obviously required that its inclusion is hardly debatable. We have already seen that Quine denies (by implication) that this constitutes an additional requirement: "The condition upon all acceptable explications of number . . . can be put . . . : any *progression*—i.e., any infinite series each of whose members has only finitely many precursors—will do nicely"(see note 3). But suppose, for example, that one chose the progression $A = a_1, a_2, a_3, \ldots a_n, \ldots$ obtained as follows. Divide the positive integers into two sequences B and C, within each sequence letting the elements come in order of magnitude. Let B (that is, $b_1, b_2, \ldots$) be the sequence of Gödel numbers of valid formulas of quantification theory, under some suitable numbering, and let C (that is, $c_1, c_2 \ldots$) be the sequence

3 It is not universally agreed that these last two parts of our account (defining the operations and defining cardinality) are indeed required for an adequate explication of number. W. V. Quine, for one, explicitly denies that anything need be done other than provide a progression to serve as the numbers. In *Word and Object* (London, 1960), pp. 262-263, he states: "The condition upon all acceptable explications of number . . . can be put . . .: *any progression*—i.e., any infinite series each of whose members has only finitely many precursors—will do nicely. Russell once held that a further condition had to be met, to the effect that there be a way of applying one's would-be numbers to the measurement of multiplicity: a way of saying that (1) There are n objects x such that Fx. This, however, was a mistake, for any progression can be fitted to that further condition. For (1) can be paraphrased as saying that the numbers less than n [Quine uses 0 as well] admit of correlation with the objects x such that Fx. This requires that our apparatus include enough of the elementary theory of relations for talk of correlation, or one-one relation; but it requires nothing special about numbers except that they form a progression." I would disagree. The explanation of cardinality—i.e., of the use of numbers for "transitive counting," as I have called it—is part and parcel of the explication of number. Indeed, if it may be excluded on the grounds Quine offers, we might as well say that there are *no* necessary conditions, since the only one he cites is hardly necessary, provided "that our apparatus contain enough of the theory of sets to contain a progression." But I will return to this point.

of positive integers which are not numbers of valid formulas of quantification theory under that numbering (in order of magnitude in each case). Now in the sequence A, for each n let $a_{2n-1} = b_n$ and $a_{2n} = c_n$. Clearly A, though a progression, is not recursive, much less primitive recursive. Just as clearly, this progression would be unusable as the numbers—and the reason is that we expect that if we know which numbers two expressions designate, we are able to calculate in a finite number of steps which is the "greater" (in this case, which one comes later in A).[4] More dramatically, if told that set b has n members, and that c has m, it should be possible to determine in a finite number of steps which has more members. Yet it is precisely that which is not possible here. This ability (to tell in a finite number of steps which of two numbers is the greater) is connected with (both transitive and intransitive) counting, since its possibility is equivalent to the possibility of generating ("saying") the numbers in order of magnitude (that is, in their order in A). You could not know that you were saying them in order of magnitude since, no recursive rule existing for generating its members, you could not know what their order of magnitude should be. This is, of course, a very strong claim. There are two questions here, both of which are interesting and neither of which could conceivably receive discussion in this paper. (1) Could a human being be a decision procedure for nonrecursive sets, or is the human organism at

[4] There is, of course, a difficulty with the notion of "knowing which numbers two expressions designate." It is the old one illustrated by the following example. Abraham thinks of a number, and so does Isaac. Call Abraham's number a and Isaac's i. Is a greater than i? I know which number a refers to : Abraham's. And similarly with i. But that brings me no closer to deciding which is the greater. This can be avoided, however, by requiring that numbers be given in canonical notation, as follows. Let the usual (recursive) definition of the numbers serve to define the *set* of "numbers," but not to establish their order. Then take the above definition of a as defining the *less-than* relation among the members of that set, thus defining the *progression*. (The fact that the nonrecursive progression that I use is a progression of *numbers* is clearly inessential to the point at issue. I use it here merely to avoid the elaborate circumlocutions that would result from doing everything set-theoretically. One could get the same effect by letting the "numbers" be formulas of quantification theory, instead of their Gödel numbers, and using the formulas autonymously.)

best a Turing machine (in the relevant respect)? If the latter, then there could not exist a human being who could generate the sequence A, much less *know* that this is what he was doing. Even if the answer to (1), however, is that a human being *could* be (act or be used as) such a decision procedure, the following question would still arise and need an answer: (2) could he *know* all truths of the form $i < j$ (in A)? And it seems that what constitutes knowledge might preclude such a possibility.

But I have digressed enough on this issue. The main point is that the "$<$" relation over the numbers must be recursive. Obviously I cannot give a rigorous proof that this is a requirement, because I cannot prove that man is at best a Turing machine. That the requirement is met by the usual "$<$" relation among numbers—the paradigm of a primitive recursive relation—and has also been met in every detailed analysis ever proposed constitutes good evidence for its correctness.[5] I am just making explicit what almost everyone takes for granted. Later in this paper, we will see that one plausible account of why this is taken for granted connects very closely with one of the views I will be urging.

So it was thus that Ernie learned that he had really been doing number theory all his life (I guess in much the same way that *our* children will learn this surprising fact about themselves if the *nouvelle vague* of mathematics teachers manages to drown them all).

It should be clear that Ernie's education is now complete. He has learned to speak with the vulgar, and it should be obvious to all that my earlier description was correct. He had at his disposal all that was needed for the concept of number. One might even say that he already possessed the concepts of number, cardinal, ordinal, and the usual operations on them, and needed only to learn a different vocabulary. It is my claim that there is nothing having to do with the task of "reducing" the concept of number to logic (or set theory) that has not been done above, or that could not be done along the lines already marked out.

[5] Needless to say, it is trivially met in any analysis that provides an effective correlation between the names of the "numbers" of the analysis and the more common names under which we know those numbers.

To recapitulate: It was necessary (1) to give definitions of "1," "number," and "successor," and "+," "×," and so forth, on the basis of which the laws of arithmetic could be derived; and (2) to explain the "extramathematical" uses of numbers, the principal one being counting—thereby introducing the concept of *cardinality* and cardinal number.

I trust that both were done satisfactorily, that the preceding contains all the elements of a correct account, albeit somewhat incompletely. None of the above was essentially new; I apologize for the tedium of expounding these details yet another time, but it will be crucial to my point that the sufficiency of the above account be clearly seen. For if it is sufficient, presumably Ernie *now* knows which sets the numbers are.

II. The Dilemma

The story told in the previous section could have been told about Ernie's friend Johnny as well. For his education also satisfied the conditions just mentioned. Delighted with what they had learned, they started proving theorems about numbers. Comparing notes, they soon became aware that something was wrong, for a dispute immediately ensued about whether or not 3 belonged to 17. Ernie said that it did, Johnny that it did not. Attempts to settle this by asking ordinary folk (who had been dealing with numbers *as* numbers for a long time) understandably brought only blank stares. In support of his view, Ernie pointed to his theorem that for any two numbers, x and y, x is less than y if and only if x belongs to y and x is a proper subset of y. Since by common admission 3 is less than 17 it followed that 3 belonged to 17. Johnny, on the other hand, countered that Ernie's "theorem" was mistaken, for given two numbers, x and y, x belongs to y if and only if y is the successor of x. These were clearly incompatible "theorems." Excluding the possibility of the inconsistency of their common set theory, the incompatibility must reside in the definitions. First "less-than." But both held that x is less than y if and only if x bears R to y. A little probing, however, revealed the source of the trouble. For Ernie, the successor under R of a number x was the set consisting of x and all the members of x, while for Johnny the successor of x was simply [x], the unit set of x—the set whose only member is x. Since for each of them 1 was the unit set of the null set, their respective progressions were

(i) [∅], [∅,[∅]], [∅,[∅],[∅,[∅]]], . . . for Ernie

and

(ii) [∅], [[∅]], [[[∅]]], . . . for Johnny.

There were further disagreements. As you will recall, Ernie had been able to prove that a set had n members if and only if it could be put into one-to-one correspondence with the set of numbers less than or equal to n. Johnny concurred. But they disagreed when Ernie claimed further that a set had n members if and only if it could be put into one-to-one correspondence with the number n itself. For Johnny, every number is single-membered. In short, their cardinality relations were different. For Ernie, 17 had 17 members, while for Johnny it had only one.[6] And so it went.

Under the circumstances, it became perfectly obvious why these disagreements arose. But what did not become perfectly obvious was how they were to be resolved. For the problem was this:

If the conclusions of the previous section are correct, then both boys have been given correct accounts of the numbers. Each was told by his father which set the set of numbers really was. Each was taught which object—whose independent existence he was able to prove—was the number 3. Each was given an account of the meaning (and reference) of number words to which no exception could be taken and on the basis of which all that we know about or do with numbers could be explained. Each was taught that some particular set of objects contained what people who used number words were really referring to. But the sets were different in each case. And so were the relations defined on these sets—including crucial ones, like cardinality and the like. But if, as I think we agreed, the account of the previous section was correct—not only as far as it went but correct in that it contained

[6] Some of their type-theoretical cousins had even more peculiar views—for to be of cardinality 5 a set had to *belong* to one of the numbers 5. I say "some of" because others did not use that definition of cardinality, or of numbers, but sided either with Ernie or with Johnny.

conditions which were both necessary and *sufficient* for any correct account of the phenomena under discussion, then the fact that they disagree on which particular sets the numbers are is fatal to the view that each number is some particular set. For if the number 3 is in fact some particular set b, it cannot be that two correct accounts of the meaning of "3"—and therefore also of its reference—assign two different sets to 3. For if it is true that for some set b, $3 = b$, then it cannot be true that for some set c, different from b, $3 = c$. But if Ernie's account is adequate in virtue of satisfying the conditions spelled out in Section I, so is Johnny's, for it too satisfies those conditions. We are left in a quandary. We have two (infinitely many, really) accounts of the meaning of certain words ("number," "one," "seventeen," and so forth) each of which satisfies what appear to be necessary and sufficient conditions for a correct account. Although there are differences between the two accounts, it appears that both are correct in virtue of satisfying common conditions. If so, the differences are incidental and do not affect correctness. Furthermore, in Fregean terminology, each account fixes the *sense* of the words whose analysis it provides. Each account must also, therefore, fix the *reference* of these expressions. Yet, as we have seen, one way in which these accounts differ is in the referents assigned to the terms under analysis. This leaves us with the following alternatives:

(A) Both are right in their contentions: each account contained conditions each of which was necessary and which were jointly sufficient. Therefore $3 = [[[\emptyset]]]$, and $3 = [\emptyset,[\emptyset],[\emptyset,[\emptyset]]]$.

(B) It is not the case that both accounts were correct; that is, at least one contained conditions which were not necessary and possibly failed to contain further conditions which, taken together with those remaining, would make a set of sufficient conditions.

(A) is, of course, absurd. So we must explore (B).

The two accounts agree in over-all structure. They disagree when it comes to fixing the referents for the terms in question. Given the identification of the numbers as some particular set of sets, the two accounts generally agree on the relations defined on that set; under both, we have what is demonstrably a recursive progression and a successor function which follows the order of that progression. Furthermore, the notions of cardinality are defined in terms of the progression, insuring that it becomes a theorem for each *n* that a set has *n* members if and only if it can be put into one-to-one correspondence with the set of numbers less than or equal to *n*. Finally, the ordinary arithmetical operations are defined for these "numbers." They do differ in the way in which cardinality is defined, for in Ernie's account the fact that the number *n* had *n* members was exploited to define the notion of having *n* members. In all other respects, however, they agree.

Therefore, if it is not the case that both $3 = [[[\emptyset]]]$ and $3 = [\emptyset,[\emptyset],[\emptyset,[\emptyset]]]$, which it surely is not, then at least one of the corresponding accounts is incorrect as a result of containing a condition that is not necessary. It may be incorrect in other respects as well, but at least that much is clear. I can distinguish two possibilities again: either all the conditions just listed, which both of these accounts share, are necessary for a correct and complete account, or some are not. Let us assume that the former is the case, although I reserve the right to discard this assumption if it becomes necessary to question it.

If all the conditions they share are necessary, then the superfluous conditions are to be found among those that are not shared. Again there are two possibilities: either at least one of the accounts satisfying the conditions we are assuming to be necessary, but which assigns a definite set to each number, is correct, or none are. Clearly no two different ones can be, since they are not even extensionally equivalent, much less intensionally. Therefore exactly one is correct or none is. But then the correct one must be the one that picks out which set of sets is *in fact* the numbers. We are now faced with a crucial problem: if there exists such a "correct" account, do there also exist arguments which will show it to be the correct one? Or does there exist a particular set of sets *b*, which is *really* the numbers, but such that there exists no argument one can give to establish that it, and not, say, Ernie's set $\mathcal{N}$, is really the numbers? It seems altogether too obvious that this latter possibility borders on the absurd. If the numbers constitute one particular set of sets, and not another, then there must be arguments to indicate which. In urging this I am not committing myself to the decidability by proof of every mathematical question—for I consider this neither a mathematical question nor one amenable to proof. The answer to the question I am raising will follow from an analysis of questions of the form "Is $n = \ldots$?" It will suffice for now to point to the difference between our question and

> Is there a greatest prime p such that $p + 2$ is also prime?

or even

> Does there exist an infinite set of real numbers equivalent with neither the set of integers nor with the set of all real numbers?

In awaiting enlightenment on the true identity of 3 we are not awaiting a proof of some deep theorem. Having gotten as far as we have without settling the identity of 3, we can go no further. We do not know what a proof of that *could* look like. The notion of "correct account" is breaking loose from its moorings if we admit of the possible existence of unjustifiable but correct answers to questions such as this. To take seriously the question "Is $3 = [[[\emptyset]]]$?" *tout court* (and not elliptically for "in Ernie's account?"), in the absence of any way of settling it, is to lose one's bearings completely. No, if such a question has an answer, there are arguments supporting it, and if there are no such arguments, then there *is* no "correct" account that discriminates among all the accounts satisfying the conditions of which we reminded ourselves a couple of pages back.

How then might one distinguish *the* correct account from all the possible ones? Is there a set of sets that has a greater claim to be the numbers than any other? Are there reasons one can offer to single out that set? Frege chose as the number 3 the extension of the concept "equivalent with some 3-membered set"; that is, for Frege a number was an equivalence class—the class of all classes equivalent with a given class. Although an appealing notion, there seems little to recommend it over, say, Ernie's. It has been argued that this is a more fitting account because number words are really class predicates, and that this account reveals that fact. The view is that in saying that there are n F's you are predicating n-hood of F, just as in saying that red is a color you are predicating colorhood of red. I do not think this is true. And neither did Frege.[7] It is certainly true that to say

(1) There are seventeen lions in the zoo

is not to predicate seventeen-hood of each individual lion. I suppose that it is also true that if there are seventeen lions in the zoo and also seventeen tigers in the zoo, the classes of lions-in-the-zoo and tigers-in-the-zoo are in a class together, though we shall return to that. It does not follow from this that (1) predicates seventeen-hood of one of those classes. First of all, the grammatical evidence for this is scanty indeed. The best one can conjure up by way of an example of the occurrence of a number word in predicative position is a rather artificial one like

(2) The lions in the zoo are seventeen.

If we do not interpret this as a statement about the ages of the beasts, we see that such statements do not predicate anything of any individual lion. One might then succumb to the temptation of analyzing (2) as the noun phrase "The lions in the zoo" followed by the verb phrase "are seventeen," where the analysis is parallel to that of

(3) The Cherokees are vanishing

where the noun phrase refers to the class and the verb phrase predicates something of that class. But the parallel is short-lived. For we soon notice that (2) probably comes into the language by deletion from

(4) The lions in the zoo are seventeen in number,

which in turn probably derives from something like

(5) Seventeen lions are in the zoo.

This is no place to explore in detail the grammar of number words. Suffice it to point out that they differ in many important

[7] Cf. Frege, *The Foundation of Arithmetic* (New York, 1950), sec. 57.

respects from words we do not hesitate to call predicates. Probably the closest thing to a genuine class predicate involving number words is something on the model of "seventeen-membered" or "has seventeen members." But the step from there to "seventeen" being itself a predicate of classes is a long one indeed. In fact, I should think that pointing to the above two predicates gives away the show—for what is to be the analysis of "seventeen" as it occurs in those phrases?

Not only is there scanty grammatical evidence for this view, there seems to be considerable evidence against it, as any scrutiny of the similarity of function among the number words and "many," "few," "all," "some," "any," and so forth will immediately reveal. The proper study of these matters will have to await another context, but the nonpredicative nature of number words can be further seen by noting how different they are from, say, ordinary adjectives, which do function as predicates. We have already seen that there are really no occurrences of number words in typical predicative position (that is, in "is (are) . . ."), the only putative cases being along the lines of (2) above, and therefore rather implausible. The other anomaly is that number words normally outrank *all* adjectives (or all other adjectives, if one wants to class them as such) in having to appear at the head of an adjective string, and not inside. This is such a strong ranking that deviation virtually inevitably results in ungrammaticalness:

(6) The five lovely little square blue tiles

is fine, but any modification of the position of "five" yields an ungrammatical string; the farther to the right, the worse.[8]

Further reason for denying the predicative nature of number words comes from the traditional first-order analysis of sentences

[8] It might be thought that constructions such as

(i) The hungry five went home

constitue counterexamples to the thesis that number words must come first in an adjective string. But they do not. For in (i) and similar cases, the number word occurs as a noun, and not as an adjective, probably deriving from

(ii) The five hungry NP_{pl} went home

by the obvious transformation, and should be understood as such. There are certain genuine counterexamples, but the matter is too complicated for discussion here.

such as (1), with which we started. For that is usually analyzed as:

$$(7)\quad (\exists x_1)\ldots(\exists x_{17})(Lx_1 \cdot Lx_2 \cdot \ldots \cdot Lx_{17} \cdot x_1 \neq x_2 \cdot x_1 \neq x_3 \cdot \ldots \cdot x_{16} \neq x_{17} \cdot (y)(Ly \supset . \, y = x_1 \vee y = x_2 \vee \ldots y = x_{17})).$$

The only predicate in (1) which remains is "lion in the zoo," "seventeen" giving way to numerous quantifiers, truth functions, variables, and occurences of "=," unless, of course, one wishes to consider these also to be predicates of classes. But there are slim grounds indeed for the view that (1) or (7) predicates seventeen-hood of the class of lions in the zoo. Number words function so much like operators such as "all," "some," and so forth, that a readiness to make class names of them should be accompanied by a readiness to make the corresponding move with respect to quantifiers, thereby proving (in traditional philosophic fashion) the existence of the one, the many, the few, the all, the some, the any, the every, the several, and the each.[9]

But then, what support *does* this view have? Well, this much: if two classes each have seventeen members, there probably exists a class which contains them both in virtue of that fact. I say "probably" because this varies from set theory to set theory. For example, this is not the case with type theory, since the two classes have both to be of the same type. But in no consistent theory is there a class of all classes with seventeen members, at least not alongside the other standard set-theoretical apparatus. The existence of the paradoxes is itself a good reason to deny to "seventeen" this univocal role of designating the class of all classes with seventeen members.

I think, therefore, that we may conclude that "seventeen" *need* not be considered a predicate of classes, and there is similarly no necessity to view 3 as the set of all triplets. This is not to deny that "is a class having three members" is a predicate of classes; but that is a different matter indeed. For that follows from all of the accounts under consideration.[10] Our present problem is to see if there is one account which can be established to the exclusion of all others, thereby settling the issue of which sets the numbers

[9] And indeed why not "I am the one who gave his all in fighting for the few against the many"?

[10] Within the bounds imposed by consistency.

really are. And it should be clear by now that there is not. Any purpose we may have in giving an account of the notion of number and of the individual numbers, other than the question-begging one of proving of the right set of sets that *it* is the set of numbers, will be equally well (or badly) served by any one of the infinitely many accounts satisfying the conditions we set out so tediously. There is little need to examine all the possibilities in detail, once the traditionally favored one of Frege and Russell has been seen not to be uniquely suitable.

Where does that leave us? I have argued that at most one of the infinitely many different accounts satisfying our conditions can be correct, on the grounds that they are not even extensionally equivalent, and therefore at least all but one, and possibly all, contain conditions that are not necessary and that lead to the identification of the numbers with some particular set of sets. If numbers are sets, then they must be *particular sets*, for each set is some particular set. But if the number 3 is really one set rather than another, it must be possible to give some cogent reason for thinking so; for the position that this is an unknowable truth is hardly tenable. But there seems to be little to choose among the accounts. Relative to our purposes in giving an account of these matters, one will do as well as another, stylistic preferences aside. There is no way connected with the reference of number words that will allow us to choose among them, *for the accounts differ at places where there is no connection whatever between features of the accounts and our uses of the words in question.* If all the above is cogent, then there is little to conclude except that any feature of an account that identifies 3 with a set is a superfluous one—and that therefore 3, and its fellow numbers, could not be sets at all.

III. Way Out

In this third and final section, I shall examine and urge some considerations that I hope will lend plausibility to the conclusion of the previous section, if only by contrast. The issues involved are evidently so numerous and complex, and cover such a broad spectrum of philosophic problems, that in this paper I can do no more than indicate what I think they are and how, in general, I think they may be resolved. I hope nevertheless that a more positive account will emerge from these considerations.

A. Identity. Throughout the first two sections, I have treated expressions of the form

$$n = s, \tag{8}$$

where n is a number expression and s a set expression as if I thought that they made perfectly good sense, and that it was our job to sort the true from the false.[11] And it might appear that I had concluded that all such statements were false. I did this to dramatize the kind of answer that a Fregean might give to the request for an analysis of number—to point up the kind of question Frege took it to be. For he clearly wanted the analysis to determine a truth value for each such identity. In fact, he wanted to determine a sense for the result of replacing s with any name or description whatsoever (while an expression ordinarily believed to name a number occupied the position of n). Given the symmetry and transitivity of identity, there were three kinds of identities satisfying these conditions, corresponding to the three kinds of expressions that can appear on the right:

(a) with some arithmetical expression on the right as well as on the left (for example, "$2^{17} = 4{,}892$," and so forth);
(b) with an expression designating a number, but not in a standard arithmetical way, as "the number of apples in the pot," or "the number of F's" (for example, 7 = the number of the dwarfs);
(c) with a referring expression on the right which is of neither of the above sorts, such as "Julius Caesar," "[[∅]]" (for example, 17 = [[[∅]]]).

The requirement that the usual laws of arithmetic follow from the account takes care of all identities of the first sort. Adding an explication of the concept of cardinality will then suffice for those

[11] I was pleased to find that several of the points in my discussion of Frege have been made quite independently by Charles Parsons in a paper entitled "Frege's Thesis that Numbers Are Objects," unpublished. I am indebted to his discussion for a number of improvements.

of kind (b). But to include those of kind (c), Frege felt it necessary to find some "objects" for number words to name and with which numbers could be identical. It was at this point that questions about which set of objects the numbers *really* were began to appear to need answering for, evidently, the simple answer "numbers" would not do. To speak from Frege's standpoint, there is a world of objects—that is, the designata or referents of names, descriptions, and so forth—in which the identity relation had free reign. It made sense for Frege to ask of *any* two names (or descriptions) whether they named the same object or different ones. Hence the complaint at one point in his argument that, thus far, one could not tell from his definitions whether Julius Caesar was a number.

I rather doubt that in order to explicate the use and meaning of number words one will have to decide whether Julius Caesar was (is?) or was not the number 43. Frege's insistence that this needed to be done stemmed, I think, from his (demonstrably) inconsistent logic (interpreted sufficiently broadly to encompass set theory). All items (names) in the universe were on a par, and the question whether two names had the same referent always presumably had an answer—yes or no. The inconsistency of the logic from which this stems is of course *some* reason to regard the view with suspicion. But it is hardly a refutation, since one might grant the meaningfulness of all identity statements, the existence of a universal set as the range of the relation, and still have principles of set existence sufficiently restrictive to avoid inconsistency. But such a view, divorced from the naïve set theory from which it stems, loses much of its appeal. I suggest, tentatively, that we look at the matter differently.

I propose to deny that all identities are meaningful, in particular to discard all questions of the form of (c) above as senseless or "unsemantical" (they are not totally senseless, for we grasp enough of their sense to explain why they are senseless). Identity statements make sense only in contexts where there exist possible individuating conditions. If an expression of the form "$x = y$" is to have a sense, it can be only in contexts where it is clear that both x and y are of some kind or category C, and that it is the conditions which individuate things *as the same C* which are operative and determine its truth value. An example might help clarify the point. If we know x and y to be lampposts (possibly the same, but nothing in the way they are designated decides the issue) we can ask if they are *the same lamppost.* It will be their color, history, mass, position, and so forth which will determine if they are indeed the same lamppost. Similarly, if we know z and w to be numbers, then we can ask if they are *the same number.* And it will be whether they are prime, greater than 17, and so forth which will decide if they are indeed the same number. But just as we cannot individuate a lamppost in terms of these latter predicates, neither can we individuate a number in terms of its mass, color, or similar considerations. What determines that something is a *particular lamppost* could not individuate it as a *particular number.* I am arguing that questions of the identity of a particular "entity" do not make sense. "Entity" is too broad. For such questions to make sense, there must be a well-entrenched predicate C, in terms of which one then asks about the identity of a *particular* C, and the conditions associated with identifying C's as *the same* C will be the deciding ones. Therefore, if for two predicates F and G there is no third predicate C which subsumes both and which has associated with it some uniform conditions for identifying two putative elements as the same (or different) C's, the identity statements crossing the F and G boundary will not make sense.[12] For example, it will make sense to ask of something x (which is in fact a chair) if it is the same . . . as y (which in fact is a table). For we can fill the blank with a predicate, "piece of furniture," and we know what it is for a and b to be the same or different pieces of furniture. To put the point differently, questions of identity contain the presupposition that the "entities" inquired about both belong to some general category. This presupposition is normally carried by the context or theory (that is, a more systematic context). To say that they are both "entities" is to make no presuppositions at all—for everything purports to be at

[12] To give a precise account, it will be necessary to explain "uniform conditions" in such a way as to rule out the obvious counterexamples generated by constructed *ad hoc* disjunctive conditions. But to discuss the way to do this would take us too far afield. I do not pretend to know the answer in any detail.

least that. "Entity," "thing," "object" are words having a role in the language; they are place fillers whose function is analogous to that of pronouns (and, in more formalized contexts, to variables of quantification).

Identity *is* id-entity, but only within narrowly restricted contexts. Alternatively, what constitutes an entity is category or theory dependent. There are really two correlative ways of looking at the problem. One might conclude that identity is systematically ambiguous, or else one might agree with Frege, that identity is unambiguous, always meaning sameness of object, but that (contra-Frege now) the notion of an *object* varies from theory to theory, category to category—and therefore that his mistake lay in failure to realize this fact. This last is what I am urging, for it has the virtue of preserving identity as a general logical relation whose application in any given well-defined context (that is, one within which the notion of object is univocal) remains unproblematic. Logic can then still be seen as the most general of disciplines, applicable in the same way to and within any given theory. It remains the tool applicable to all disciplines and theories, the difference being only that it is left to the discipline or theory to determine what shall count as an "object" or "individual."

That this is not an implausible view is also suggested by the language. Contexts of the form "the same *G*" abound, and indeed it is in terms of them that identity should be explained, for what will be counted as the same *G* will depend heavily on *G*. The same *man* will have to be an individual man; "the same *act*" is a description that can be satisfied by many individual acts, or by only one, for the individuating conditions for acts make them sometimes types, sometimes tokens. Very rare in the language are contexts open to (satisfiable by) any kind of "thing" whatsoever. There are some—for example, "Sam referred to . . .," "Helen thought of . . ."—and it seems perfectly all right to ask if what Sam referred to on some occasion was what Helen thought of. But these contexts are very few, and they all seem to be intensional, which casts a referentially opaque shadow over the role that identity plays in them.

Some will want to argue that identities of type (c) are not senseless or unsemantical, but simply false—on the grounds that the distinction of categories is one that cannot be drawn. I have only the following argument to counter such a view. It will be just as hard to explain how one *knows* that they are false as it would be to explain how one knows that they are senseless, for normally we know the falsity of some identity "$x = y$" only if we know of x (or y) that it has some characteristic that we know y (or x) *not* to have. I know that $2 \neq 3$ because I know, for example, that 3 is odd and 2 is not, yet it seems clearly wrong to argue that we know that $3 \neq [[[\emptyset]]]$ because, say, we know that 3 has no (or seventeen, or infinitely many) members while $[[[\emptyset]]]$ has exactly one. We know no such thing. We do not know that it does. But that does not constitute knowing that it does not. What is enticing about the view that these are all false is, of course, that they hardly seem to be open questions to which we may find the answer any day. Clearly, all the evidence is in; if no decision is possible on the basis of it, none will ever be possible. But for the purposes at hand the difference between these two views is not a very serious one. I should certainly be happy with the conclusion that all identities of type (c) are either senseless or false.

B. Explication and Reduction. I would like now to approach the question from a slightly different angle. Throughout this paper, I have been discussing what was substantially Frege's view, in an effort to cast some light on the meaning of number words by exposing the difficulties involved in trying to determine which objects the numbers really are. The analyses we have considered all contain the condition that numbers are sets, and that therefore each individual number is some individual set. We concluded at the end of Section II that numbers could not be sets at all—on the grounds that there are no good reasons to say that any particular number is some particular set. To bolster our argument, it might be instructive to look briefly at two activities closely related to that of stating that numbers *are* sets—those of explication and reduction.

In putting forth an explication of number, a philosopher may have as part of his explication the statement that $3 = [[[\emptyset]]]$. Does it follow that he is making the kind of mistake of which I accused Frege? I think not. For there is a difference between *asserting* that 3 *is* the set of all triplets and *identifying* 3 with that set, which last is what might be done in the context of some explication. I certainly do not wish what I am arguing in this paper to militate against identifying 3 with anything you like. The difference lies in that, normally, one who identifies 3 with some particular set does so for the purpose of presenting some theory and does not claim that he has *discovered* which object 3 really is. We might want to know whether some set (and relations and so forth) would do as number surrogates. In investigating this it would be entirely legitimate to state that making such an identification, we can do with that set (and those relations) what we now do with the numbers. Hence we find Quine saying:

Frege dealt with the question "What is a number?" by showing how the work for which the objects in question might be wanted could be done by objects whose nature was presumed to be less in question.[13]

Ignoring whether this is a correct interpretation of Frege, it is clear that someone who says this would not claim that, since the answer turned out to be "Yes," it is now clear that numbers were really sets all along. In such a context, the adequacy of some system of objects to the task is a very real question and one which can be settled. Under our analysis, *any* system of objects, sets or not, that forms a recursive progression must be adequate. It is therefore obvious that to discover that a system will do cannot be to discover which objects the numbers are . . . . Explication, in the above reductionistic sense, is therefore neutral with respect to the sort of problem we have been discussing, but it does cast some sobering light on what it is to be an individual number.

There is another reason to deny that it would be legitimate to use the reducibility of arithmetic to set theory as a reason to assert that numbers are really sets after all. Gaisi Takeuti has shown that the Gödel-von Neumann-Bernays set theory is in a strong sense *reducible to* the theory of ordinal numbers less than the least inaccessible number.[14] No wonder numbers are sets; sets are really (ordinal) numbers, after all. *But now, which is really which?*

[13] Quine, *op. cit.*, p. 262.

[14] Takeuti, "Construction of the Set Theory from the Theory of Ordinal Numbers," *Journal of the Mathematical Society of Japan*, 6 (1954).

These brief comments on reduction, explication, and what they might be said to achieve in mathematics lead us back to the quotation from Richard Martin which heads this paper. Martin correctly points out that the mathematician's interest stops at the level of structure. If one theory can be modeled in another (that is, reduced to another) then further questions about whether the individuals of one theory are really those of the second just do not arise. In the same passage, Martin goes on to point out (approvingly, I take it) that the philosopher is not satisfied with this limited view of things. He wants to know more and does ask the questions in which the mathematician professes no interest. I agree. He does. And mistakenly so. It will be the burden of the rest of this paper to argue that such questions miss the point of what arithmetic, at least, is all about.

C. Conclusion: Numbers and Objects. It was pointed out above that any system of objects, whether sets or not, that forms a recursive progression must be adequate. But this is odd, for any recursive set can be arranged in a recursive progression. So what matters, really, is not any condition on the *objects* (that is, on the set) but rather a condition on the relation under which they form a progression. To put the point differently—and this is the crux of the matter—that any recursive sequence whatever would do suggests that what is important is not the individuality of each element but the structure which they jointly exhibit. This is an extremely striking feature. One would be led to expect from this fact alone that the question of whether a particular "object"—for example, [[[∅]]]—would do as a replacement for the number 3 would be pointless in the extreme, as indeed it is. "Objects" do not do the job of numbers singly; the whole system performs the job or nothing does. I therefore argue, extending the argument that led to the conclusion that numbers could not be sets, that numbers could not be objects at all; for there is no more reason to identify any individual number with any one particular object than with any other (not already known to be a number).

The pointlessness of trying to determine which objects the numbers are thus derives directly from the pointlessness of asking the question of any individual number. For arithmetical purposes the properties of numbers which do not stem from the relations they bear to one another in virtue of being arranged in a progression are of no consequence whatsoever. But it would be only these properties that would single out a number as this object or that.

Therefore, numbers are not objects at all, because in giving the properties (that is, necessary and sufficient) of numbers you merely characterize an *abstract structure*—and the distinction lies in the fact that the "elements" of the structure have no properties other than those relating them to other "elements" of the same structure. If we identify an abstract structure with a system of relations (in intension, of course, or else with the set of all relations in extension isomorphic to a given system of relations), we get arithmetic elaborating the properties of the "less-than" relation, or of all systems of objects (that is, *concrete* structures) exhibiting that abstract structure. That a system of objects exhibits the structure of the integers implies that the elements of that system have some properties not dependent on structure. It must be possible to individuate those objects independently of the role they play in that structure. But this is precisely what cannot be done with the numbers. To *be* the number 3 is no more and no less than to be preceded by 2, 1, and possibly 0, and to be followed by 4, 5, and so forth. And to *be* the number 4 is no more and no less than to be preceded by 3, 2, 1, and possibly 0, and to be followed by.... *Any* object can *play the role of* 3; that is, any object can be the third element in some progression. What is peculiar to 3 is that it defines that role—not by being a paradigm of any object which plays it, but by representing the relation that any third member of a progression bears to the rest of the progression.

Arithmetic is therefore the science that elaborates the abstract structure that all progressions have in common merely in virtue of being progressions. It is not a science concerned with particular objects—the numbers. The search for which independently identifiable particular objects the numbers really are (sets? Julius Caesars?) is a misguided one.

On this view many things that puzzled us in this paper seem to fall into place. Why so many interpretations of number theory are possible without any being uniquely singled out becomes obvious: there is no unique set of objects that are the numbers. Number theory is the elaboration of the properties of *all* structures of the order type of the numbers. The number words do not have single referents. Furthermore, the reason identification of numbers with objects works wholesale but fails utterly object by object is the fact that the theory is elaborating an abstract structure and not the properties of independent individuals, any one of which could be characterized without reference to its relations to the rest. Only when we are considering a particular sequence as being, not the numbers, but *of the structure of the numbers* does the question of which element is, or rather *corresponds to*, 3 begin to make any sense.

Slogans like "Arithmetic is about numbers," "Number words refer to numbers," when properly urged, may be interpreted as pointing out two quite distinct things: (1) that number words are not names of special nonnumerical entities, like sets, tomatoes, or Gila monsters; and (2) that a purely formalistic view that fails to assign any meaning whatsoever to the statements of number theory is also wrong. They need not be incompatible with what I am urging here.

This last formalism is too extreme. But there is a modified form of it, also denying that number words are names, which constitutes a plausible and tempting extension of the view I have been arguing. Let me suggest it here. On this view the sequence of number words is just that—a sequence of words or expressions with certain properties. There are not two kinds of things, numbers and number words, but just one, the words themselves. Most languages contain such a sequence, and any such sequence (of words or terms) will serve the purposes for which we have ours, provided it is recursive in the relevant respect. In counting, we do not correlate sets with initial segments of the numbers as extralinguistic entities, but correlate sets with initial segments of the sequence of number *words*. The central idea is that this recursive sequence is a sort of yardstick which we use to measure sets. Questions of the identification of the referents of number words should be dismissed as misguided in just the way that a question about the referents of the parts of a ruler would be seen as misguided. Although any sequence of expressions with the proper structure would do the job for which we employ our present number words, there is still some reason for having one, relatively uniform, notation: ordinary communication. Too many sequences in common use would make it necessary for us to learn too many different equivalences. The usual objection to such an account—that there is a distinction between numbers and number words which it fails to make will, I think, not do. It is made on the grounds that "two," "*zwei*," "*deux*," "2" are all supposed to "stand for" the same number but yet are *different* words (one of them not a word at all). One can mark the differences among the expressions in question, and the similarities as well, without conjuring up some extralinguistic objects for them to name. One need only point to the similarity of function: within any numbering system, what will be important will be what place in the system any particular expression is used to mark. All the above expressions share this feature with one another—and with the binary use of "10," but not with its decimal employment. The "ambiguity" of "10" is thus easily explained. Here again we see the series-related character of individual numbers, except that it is now mapped a little closer to home. One cannot tell what number a particular expression represents without being given the sequence of which it forms a part. It will then be from its place in that sequence—that is, from its relation to other members of the sequence, *and from the rules governing the use of the sequence in counting*—that it will derive its individuality. It is for this last reason that I urged, contra Quine, that the account of cardinality must explicitly be included in the account of number (see note 3).

Furthermore, other things fall into place as well. The requirement, discussed in Section I, that the "less-than" relation be recursive is most easily explained in terms of a recursive notation. After all, the whole theory of recursive functions makes most sense when viewed in close connection with notations rather than with extralinguistic objects. This makes itself most obvious in three places: the development of the theory by Post systems, by Turing machines, and in the theory of constructive ordinals, where the concern is frankly with recursive notations for ordinals. I do not see why this should not be true of the finite ordinals as well. For a set of *numbers* is recursive if and only if a machine of a particular sort could be programmed to generate them in order of magnitude—that is, to generate the standard or canonical notations for those numbers following the (reverse) order of the "less-than" relation. If that relation over the notation were not recursive, the above theorem would not hold.

It also becomes obvious why every analysis of number ever presented has had a recursive "less-than" relation. If what we are generating is a notation, the most natural way for generating it is by giving recursive rules for getting the next element from any element you may have—and you would have to go far out of your way (and be slightly mad) to generate the notation and then define "less than" as I did on pages 51-52, above, in discussing the requirement of recursiveness.

Furthermore, on this view, we learn the elementary arithmetical operations as the cardinal operations on small sets, and extend them by the usual algorithms. Arithmetic then becomes cardinal arithmetic at the earlier levels in the obvious way, and the more advanced statements become easily interpretable as *projections* via truth functions, quantifiers, and the recursive rules governing the operations. One can therefore be this sort of formalist without denying that there is such a thing as arithmetical truth other than derivability within some given system. One can even explain what the ordinary formalist apparently cannot—why these axioms were chosen and which of two possible consistent extensions we should adopt in any given case.

But I must stop here. I cannot defend this view in detail without writing a book. To return in closing to our poor abandoned children, I think we must conclude that their education was badly mismanaged—not from the mathematical point of view, since we have concluded that there is no mathematically significant difference between what they were taught and what ordinary mortals know, but from the philosophical point of view. They think that numbers are really sets of sets while, if the truth be known, there are no such things as numbers; which is not to say that there are not at least two prime numbers between 15 and 20.[15]

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[15] I am indebted to Paul Ziff for his helpful comments on an earlier draft of this paper.